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Hausdorff Dimension of Kuperberg Minimal Sets

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Introduction

- Kuperberg's flow is the flow of a C^∞ aperiodic vector field on a three-manifold called a *plug*.
- This flow preserves a unique minimal set with a fractal structure.
- Problem: What is the Hausdorff dimension of this minimal set?

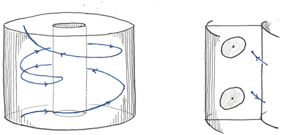
History: Seifert's conjecture

- Seifert 1950: Does every nonsingular vector field on the three-sphere S^3 have a periodic orbit?
- Wilson 1966: On any three-manifold there exists a nonsingular C^∞ vector field with only 2 periodic orbits.
  - These orbits are contained inside a *plug*.
  - The plug is inserted to break periodic orbits outside the plug.
- Schweitzer 1974: There exists a C^1 plug with no periodic orbits.
- Harrison 1988: Constructed a C^2 aperiodic plug.
- Kuperberg 1994: Constructed a C^∞ aperiodic plug.
  - Modified Wilson's construction using *self-insertion*.

The Wilson Plug

Wilson's plug is a three-manifold with boundary, supporting a smooth vector field $\mathcal{W}$ defining a flow ϕ_t with three orbit types:

- Large radius:



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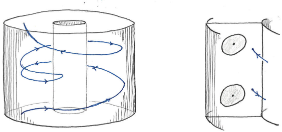
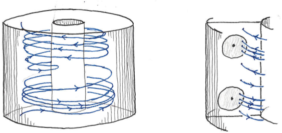


- Radius close to 2:



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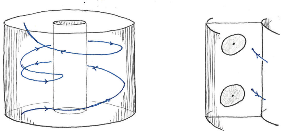

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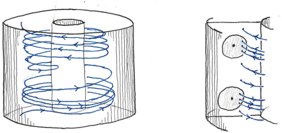
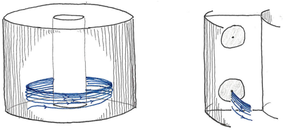
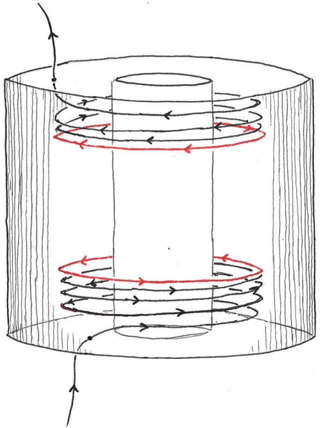


- Radius $= 2$:



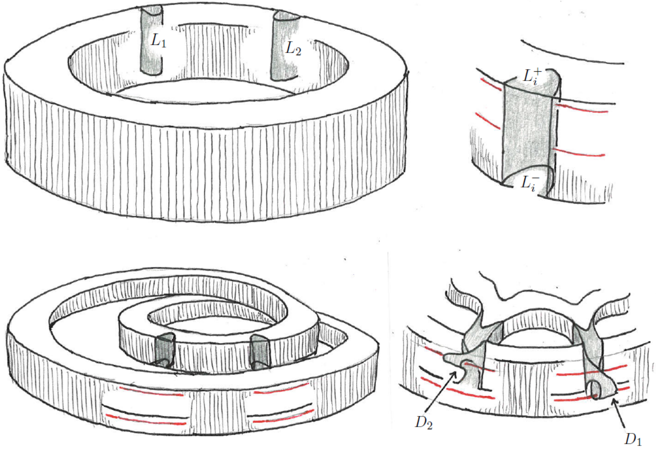
Wilson's minimal set



Two periodic orbits

Kuperberg's plug

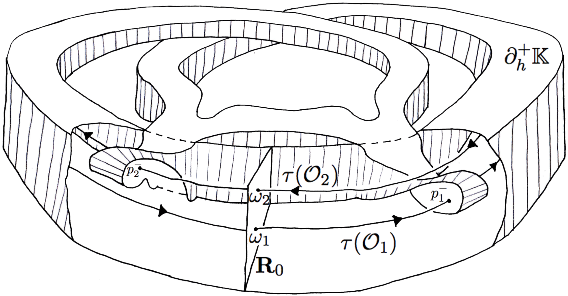
The Kuperberg plug is the Wilson plug with self-insertions. The resulting plug K inherits a vector field $\mathcal{K}$ with flow ψ_t.



Dynamics of the Kuperberg flow

Kuperberg (1994)

The C^∞ vector field $\mathcal{K}$ has no closed orbits.



The Kuperberg plug

The Kuperberg minimal set

- Ghys (1995): conjectured that K has a unique minimal set $\mathcal{M}$ with topological dimension 2.
- Hurder and Rechtman (2016):
  - $\mathcal{M} \subset K$ is nontrivial.
  - $\mathcal{M}$ is a surface lamination with radial Cantor transversal of Lebesgue measure zero.
- To estimate the Hausdorff dimension of $\mathcal{M}$ we model the transverse Cantor set as the attractor of a *conformal graph-directed pseudo-Markov system*.

Iterated Function Systems

A collection $S = \{\phi_i : X \to X\}_{i \in I}$ of injective contractions of a compact metric space X is an *IFS*. For $\omega \in I^n$, denote

$$\phi_\omega = \omega_1 \circ \cdots \circ \omega_n$$

Then $J = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \phi_\omega(X)$ is the *limit set* of S.

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- J is invariant under S.
- If S satisfies the *open set condition* and the *bounded distortion property*, then J is a Cantor set.
- Nesting condition: $\phi_{\omega,i}(X) \subset \phi_\omega(X)$.
- If $X \subset \mathbb{R}$ and ϕ_i are $C^{1+\alpha}$, then S has bounded distortion.

Topological Pressure of an IFS

Each IFS $S = \{\phi_i : X \to X\}_{i \in I}$ has an associated *topological pressure*:

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \|\phi'_\omega\|^t$$

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Theorem (Bowen 1979)

Let J be the limit set of an IFS, and $s = \dim_H(J)$. Then s is the unique solution of $P(s) = 0$.

Dimension theory of limit sets

Thermodynamic formalism:

- Mathematical formulation of equilibrium statistical mechanics developed by Sinai and Ruelle.
- Idea: study space of probability measures on phase space, with good ergodic properties.
- Hausdorff dimension estimates involve probability measures (Frostman's lemma).

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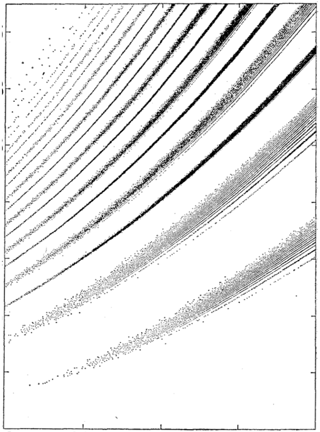
Generalizations of Bowen's theorem:

- Mauldin and Urbański (1996) extended to graph-directed constructions on infinite alphabet.
- Stratmann and Urbański (2007) extended to conformal pseudo-Markov systems (CPMS).
- Barreira, Pesin and Weiss (1996) developed a non-additive theormodynamic formalism.

Kuperberg's minimal set



Cross-section of Kuperberg minimal set

- There exists a curve γ transverse to the flow so that

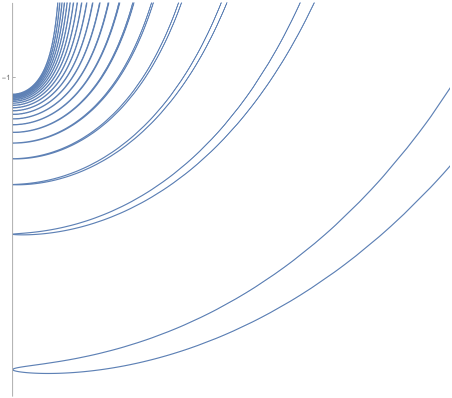
$$\mathcal{M} = \overline{\bigcup_{-\infty<t<\infty} \psi_t(\gamma)}.$$

- We can decompose $\mathcal{M}$ by level of insertion:

$$\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$$

Each $\mathcal{M}_n$ forms a *propeller* winding around the plug.

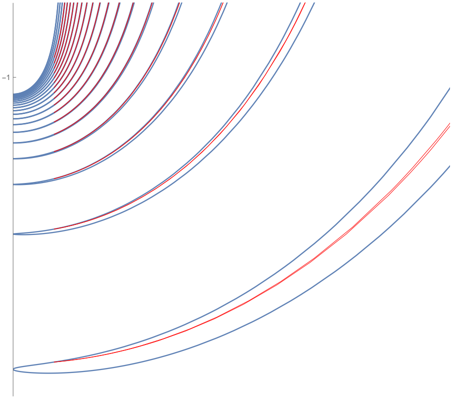
Level-one propeller $\mathcal{M}_1$



Cross-section of $\mathcal{M}_1$

- Curves are generated the holonomy pseudogroup of the Wilson flow.
- Propeller $\mathcal{M}_1$ bounds a family of closed regions R_i.
- The pseudogroup contracts R_i in the radial direction.
- Infinite returns of the propeller implies symbolic dynamics on an infinite alphabet.

Level-two propeller P_2



Cross-sections of $\mathcal{M}_1$ and $\mathcal{M}_2$

- Curves are generated by composition of Wilson pseudogroup with one insertion.
- Propeller $\mathcal{M}_2$ bounds a family of closed regions $R_{i,j}$.
- Nesting property: $R_{i,j} \subset R_j$

Pressure for pseudogroups

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in A_n \subset I^n} \|\phi'_\omega\|^t$$

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- A_n is defined by the symbolic dynamics of the pseudogroup.
- Necessary to determine growth of derivatives ϕ'_ω of the holonomy maps.
- Bounds on ϕ'_ω lead to *generalized pressure functions*.
- To estimate $P(t) = 0$, we find zeros of generalized pressure functions.

Dimension estimates on $\mathcal{M}$

Theorem (I.)

Let $C \subset [0,1]$ be the transverse Cantor set of $\mathcal{M}$.

- There exists a conformal graph-directed pseudo-Markov function system on $[0,1]$ with limit set C.
- $s = \dim_H(C)$ is the unique root of a dynamically defined pressure function.
- $0.544 \le \dim_H(C) \le 0.863$.

Corollary: $2.544 \le \dim_H(\mathcal{M}) \le \ 2.863$.

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Question: Can we use similar thermodynamic formalism to find dimension estimates for other codimension-one attractors?

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